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NON-NEGATIVE CURVATURE, ELLIPTIC GENUS AND UNBOUNDED PONTRYAGIN NUMBERS

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Abstract We discuss the cobordism type of spin manifolds with non-negative sectional curvature. We show that in each dimension $4k \geqslant 12$, there are infinitely many cobordism types of simply connected and non-negatively curved spin manifolds. Moreover, we raise and analyse a question about possible cobordism obstructions to non-negative curvature.

Keywords: non-negative curvature; spin manifolds; elliptic genus; Pontryagin numbers

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1. Introduction

Finding obstructions to non-negative or positive curvature on closed manifolds has a long tradition in Riemannian geometry. In the present article, we want to deal with the question of which rational cobordism invariants can be seen as obstructions to non-negative curvature. One such obstruction is the signature, which is bounded on connected, non-negatively curved and oriented manifolds by Gromov's Betti number theorem.

Normalizing the diameter and imposing an additional upper curvature bound restricts by Chern–Weil theory the Pontryagin numbers, and therefore the possible oriented cobordism classes, to finitely many possibilities. Nevertheless, Dessai and Tuschmann [4] proved that in all dimensions $4k \geqslant 8$, there are infinitely many oriented cobordism types of simply connected and non-negatively curved manifolds. We generalize this result to spin manifolds of non-negative sectional curvature.

Theorem 1.1. *In every dimension $4k \geqslant 12$, there are infinitely many cobordism types of simply connected, closed spin manifolds of non-negative sectional curvature.*

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Moreover, Kotschick [7] used slight generalizations of the examples given by Dessai and Tuschmann to show that every linear combination of Pontryagin numbers that is not a multiple of the signature is unbounded on oriented manifolds of non-negative sectional curvature.

We emphasize that the spin case is more difficult to treat, since there are index-theoretic obstructions. In fact, for spin manifolds with non-negative curvature the $\hat{A}$-genus, which is the index of the Dirac operator, vanishes by a Lichnerowicz-type argument (cf. [8] in a more general setting). It follows that the lower bound on the dimension in Theorem 1.1 is optimal, since in dimensions 4 and 8, every Pontryagin number is a linear combination of the signature and the $\hat{A}$-genus.

Both the signature and the $\hat{A}$-genus can be seen as the first coefficient of different expansions of the elliptic genus [5]. We recall that the elliptic genus $\phi(M)$ of a spin manifold M^{4k} is a modular function, which takes the value of the signature in one of its cusps. In the other cusp, the elliptic genus admits the q-expansion

$$\begin{aligned}\phi(M) &= q^{-k/2} \cdot \hat{A}\left(M; \bigotimes_{\substack{n \text{ odd} \\ n\geqslant 1}}^{\infty} \Lambda_{-q^n} T_{\mathbb{C}}M \otimes \bigotimes_{\substack{n \text{ even} \\ n\geqslant 1}}^{\infty} S_{q^n} T_{\mathbb{C}}M\right) \\ &= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M; T_{\mathbb{C}}M)q \pm \cdots) \in q^{-k/2}\mathbb{Z}[q].\end{aligned}$$

This expansion can be taken as a definition for the elliptic genus.

One might ask whether the elliptic genus is constant on spin manifolds of non-negative sectional curvature. For positive sectional curvature, this question was raised by Dessai [3]. To our knowledge, the question is still open.

Some evidence for a positive answer to this question is provided by the following results. First, the constancy of the elliptic genus has been shown by Hirzebruch and Slodowy [6] in the case of homogeneous spaces. For biquotients, Singhof [11] gave some partial results. Moreover, several results were obtained on the vanishing of the coefficients of the elliptic genus in the context of isometric torus actions and positive sectional curvature by Dessai [2, 3] and the second author of this paper [13].

Since it is not evident whether the elliptic genus obstructs non-negative curvature, we would like to raise the following question, which can be thought of as a direct analogue of Kotschick's result for spin manifolds, where we replace the signature by the elliptic genus.

Question 1.2. Let $f : \Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ be a linear combination of Pontryagin numbers, which is not contained in the span of the coefficients of the elliptic genus. Is f unbounded on connected, non-negatively curved spin $4k$-manifolds?

A positive answer to this question would imply that the elliptic genus is the only possible obstruction to non-negative curvature on spin manifolds from the point of view of rational oriented cobordism. Here, we prove that Question 1.2 admits a positive answer in dimensions up to 20.

Theorem 1.3. *For $k \leqslant 5$, every linear combination $f : \Omega_{4k}^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ of Pontryagin numbers that is not contained in the span of the coefficients of the elliptic genus* *is unbounded on simply connected, closed spin $4k$-manifolds of non-negative sectional curvature.*

In dimensions 4 and 8, this theorem is trivial, since any linear combination of Pontryagin numbers in these dimensions lies in the span of the signature and the $\hat{A}$-genus, and thus also in the span of the coefficients of the elliptic genus. Note that the proof of Kotschick's theorem involves the construction of a basis sequence for the rational cobordism ring consisting of non-negatively curved manifolds. In the spin case treated here, this is not possible, since in dimension 4, any non-negatively curved spin manifold is rationally nullbordant, owing to the only Pontryagin number $p_1[M^4]$ being a multiple of the $\hat{A}$-genus.

We will prove these theorems by computing the cobordism type of certain spin projective bundles over complex projective spaces.

The paper is structured as follows. In § 2 we introduce the relevant families of projective bundles and discuss the cobordism type as well as the curvature properties. In § 3 we give the proofs of the theorems.

2. Projective bundles over complex projective spaces

For the proof of Theorem 1.3, we need to construct some non-negatively curved families of spin manifolds, whose Pontryagin numbers are mutually distinct. For this purpose we consider the projectivization of complex vector bundles of rank $2k+2$ over the base $\mathbb{C}\mathrm{P}^{2l+1}$. In our case, the vector bundles decompose into a sum of complex line bundles. This construction yields a suitable family for the proof of Theorem 1.3.

2.1. Construction of the families

We start with a complex vector bundle E of rank $2k+2$ over $\mathbb{C}\mathrm{P}^{2l+1}$. Let

$$c(E) = 1 + c_1(E) + \cdots + c_{2k+2}(E) \in H^*(\mathbb{C}\mathrm{P}^{2l+1}; \mathbb{Z})$$

be the total Chern class of the vector bundle E. We take the projectivization $P(E)$ with respect to E and obtain a fibre bundle

$$\mathbb{C}\mathrm{P}^{2k+1} \hookrightarrow P(E) \twoheadrightarrow \mathbb{C}\mathrm{P}^{2l+1}.$$

It follows from the Leray–Hirsch theorem that the cohomology ring $H^*(P(E); \mathbb{Z})$ is generated as a free $H^*(\mathbb{C}\mathrm{P}^{2l+1}; \mathbb{Z})$ module by an element $a \in H^2(P(E); \mathbb{Z})$, subject to the following relation

$$a^{2k+2} + a^{2k+1}c_1(E) + \cdots + c_{2k+2}(E) = 0.$$

For the notation, we fix $b \in H^2(\mathbb{C}\mathrm{P}^{2l+1}; \mathbb{Z})$ to be a generator.

Next we are concerned with the spin structures of $P(E)$. We recall that a closed oriented manifold is spin if and only if its second Stiefel–Whitney class vanishes. The latter constitutes a homotopy invariant, and we may compute it via the Wu formula [**9**]. Alternatively, one could apply the techniques of Borel and Hirzebruch [**1**] to determine the Stiefel–Whitney classes.

Lemma 2.1. *$P(E)$ is spin if and only if $c_1(E)$ is even.*

Proof. We need to determine the second Wu class $v_2 \in H^2(P(E);\mathbb{Z}_2)$, which is uniquely characterized by the relation

$$\langle v_2 \cup x, \mu_{P(E)}\rangle = \langle \mathrm{Sq}^2(x), \mu_{P(E)}\rangle,$$

where $\mu_{P(E)}$ is the fundamental class and $x \in H^*(P(E);\mathbb{Z}_2)$ is any element. The only relevant cohomology group $H^{4k+4l+2}(P(E);\mathbb{Z}_2)$ is generated over $\mathbb{Z}_2$ by the two elements $a^{2k}b^{2l+1}$ and $a^{2k+1}b^{2l}$. We use the Cartan formula to compute the Steenrod squares

$$\mathrm{Sq}^2(a^{2k}b^{2l+1}) = 0 \quad \text{and} \quad \mathrm{Sq}^2(a^{2k+1}b^{2l}) = c_1(E)a^{2k+1}b^{2l}.$$

Thus, the second Wu class vanishes if and only if $c_1(E)$ is even, and so does the second Stiefel–Whitney class by the Wu formula. □

We recall a general recipe for the computation of the Pontryagin classes of $P(E)$. Our approach is based on the techniques of Borel and Hirzebruch. First, we observe that the fibre bundle structure of $P(E)$ induces a splitting of the tangent bundle

$$TP(E) = \pi^* T\mathbb{C}\mathrm{P}^{2l+1} \oplus \eta_E, \tag{2.1}$$

where $\pi^* T\mathbb{C}\mathrm{P}^{2l+1}$ is the pullback bundle induced by the projection and η_E is the complex bundle along the fibres. Following [**1**, § 15.1, p. 515] the Chern classes of η_E are given by

$$c(\eta_E) = \sum_{i=0}^{2k+2} (1+a)^{2k+2-i} c_i(E).$$

From here, one can easily deduce the Pontryagin classes of η_E and, in view of the splitting (2.1), the Pontryagin classes of $P(E)$ can be determined via the product formula.

Following these general considerations we turn our attention to Theorem 1.3 and give explicit families for the relevant dimensions. In dimension 12 we consider complex vector bundles $E^4 \to \mathbb{C}\mathrm{P}^3$ of the type

$$E^4 = (c \cdot \gamma^1) \oplus \epsilon^3 \quad \text{for } c \in \mathbb{Z},$$

where γ^1 denotes the dual Hopf bundle, $c \cdot \gamma^1$ is the c-fold tensor product of γ^1 and ϵ^3 is the trivial vector bundle of rank 3 over $\mathbb{C}\mathrm{P}^3$. The total Chern class is then given by

$$c(E) = 1 + c_1(E) = 1 + c \cdot b \in H^*(\mathbb{C}\mathrm{P}^3;\mathbb{Z}).$$

As before, we take the projectivization, which we write as X_c^{12}. In view of the recipe, we compute the Pontryagin numbers of X_c^{12}.

Lemma 2.2. *The Pontryagin numbers of X_c^{12} are given by*

$$p_1^3[X_c^{12}] = -8c^3, \quad p_1p_2[X_c^{12}] = -6c^3, \quad p_3[X_c^{12}] = -c^3.$$

In dimension 16 we take complex vector bundles $E^4 \to \mathbb{C}\mathrm{P}^5$ of the type

$$E^4 = (c\cdot\gamma^1) \oplus (2c\cdot\gamma^1) \oplus (-3c\cdot\gamma^1) \oplus \epsilon^1 \quad \text{for } c \in \mathbb{Z}.$$

Therefore, the total Chern class is given by

$$c(E) = 1 - 7c^2\cdot b^2 - 6c^3\cdot b^3 \in H^*(\mathbb{C}\mathrm{P}^5;\mathbb{Z}).$$

In particular, the first Chern class vanishes. As before, we projectivize these bundles to obtain a family Y_c^{16} of $\mathbb{C}\mathrm{P}^3$ bundles over $\mathbb{C}\mathrm{P}^5$. The computation of the Pontryagin numbers is carried out according to the recipe.

Lemma 2.3. *The Pontryagin numbers of Y_c^{16} are given by*

$$p_1^4[Y_c^{16}] = 768c^3(12+56c^2), \quad p_1^2p_2[Y_c^{16}] = 384c^3(15+56c^2),$$

$$p_1p_3[Y_c^{16}] = 48c^3(42+56c^2), \quad p_2^2[Y_c^{16}] = 144c^3(24+56c^2), \quad p_4[Y_c^{16}] = 288c^3.$$

Our family of 20-dimensional examples is similar to the one in dimension 12. We take the rank 4 complex vector bundles

$$E^4 = (c\cdot\gamma^1) \oplus \epsilon^3 \quad \text{for } c \in \mathbb{Z}$$

over $\mathbb{C}\mathrm{P}^7$. We denote the projectivizations as Z_c^{20}.

Lemma 2.4. *The Pontryagin numbers of Z_c^{20} are given by*

$$\begin{aligned}
p_1^5[Z_c^{20}] &= -64c^3(3c^4+30c^2+80), & p_1^3p_2[Z_c^{20}] &= -2c^3(39c^4+480c^2+1456),\\
p_1^2p_3[Z_c^{20}] &= -3c^3(3c^4+80c^2+352), & p_1p_2^2[Z_c^{20}] &= -c^3(27c^4+456c^2+1616),\\
p_1p_4[Z_c^{20}] &= -8c^3(3c^2+29), & p_2p_3[Z_c^{20}] &= -c^3(3c^4+96c^2+580),\\
p_5[Z_c^{20}] &= -28c^3.
\end{aligned}$$

Finally, it is important to note that the elliptic genus vanishes on our families X_c^{12}, Y_c^{16} and Z_c^{20}. This follows from a result by Ochanine [10] stating that the elliptic genus is multiplicative in spin fibre bundles.

2.2. Non-negative curvature on the families

Next, we show that the examples we will use to prove the theorems admit a metric of non-negative curvature. We do so in a slightly more general setting.

Let E be a complex vector bundle of rank k over the base $B = \mathbb{C}\mathrm{P}^l$ and assume it decomposes as a Whitney sum $E = \bigoplus_{i=1}^k \gamma_i$ of complex line bundles γ_i. Let P be a principal T^k bundle such that $E = P \times_{\mathrm{T}^k} \mathbb{C}^k = (P \times \mathbb{C}^k)/\mathrm{T}^k$. Using a theorem of Stewart [12] it is possible to lift the standard action of $\mathrm{SU}(l+1)$ on $\mathbb{C}\mathrm{P}^l$ to P. Then, P is a homogeneous space $P = (\mathrm{SU}(l+1) \times \mathrm{T}^k)/\rho(\mathrm{U}(l))$ where the first component ρ_1 of ρ is a standard embedding, such that $\mathbb{C}\mathrm{P}^l = \mathrm{SU}(l+1)/\rho_1(\mathrm{U}(l))$ and ρ_2 depends on the principal torus bundle.

Since $\rho_2|_{\mathrm{SU}(l)}$ has to be trivial, we obtain

$$\hat{P} = \mathrm{SU}(l+1)/\rho(\mathrm{SU}(l)) \cong \mathrm{S}^{2l+1} \times \mathrm{T}^k.$$

Let $\hat{E} = \hat{P} \times_{\mathrm{T}^k} \mathbb{C}^k$. For the associated sphere bundles we get $\mathrm{S}(\hat{E}) = \hat{P} \times_{\mathrm{T}^k} \mathrm{S}^{2k-1} \cong \mathrm{S}^{2l+1} \times \mathrm{S}^{2k-1}$ and $\mathrm{S}(E) = \mathrm{S}(\hat{E})/\mathrm{S}^1$. So the projectivized bundle $\mathrm{P}(E)$ is a quotient of $\mathrm{S}^{2l+1} \times \mathrm{S}^{2k-1}$ by a free, isometric T^2-action and therefore carries a metric of non-negative curvature.

3. Proofs of the theorems

We combine the topological and geometric ingredients to prove Theorem 1.3. It follows from the modular properties of the elliptic genus that its coefficients span a $(k+1)$-dimensional subspace of the dual space of $\Omega_{8k}^{\mathrm{SO}} \otimes \mathbb{Q}$ and $\Omega_{8k+4}^{\mathrm{SO}} \otimes \mathbb{Q}$, respectively. For the proof it suffices to show that the remaining linear combinations of Pontryagin numbers are unbounded on our families.

Proof of Theorem 1.3 for $k = 3$. Let $f : \Omega_{12}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ be a linear combination of Pontryagin numbers that is not contained in the span of the coefficients of the elliptic genus. From Thom's work, it is well known that $\Omega_{12}^{\mathrm{SO}} \otimes \mathbb{Q}$ is a three-dimensional $\mathbb{Q}$-vector space. Using

$$\mathrm{sign}(M) = \frac{1}{945}(62p_3[M] - 13p_1p_2[M] + 2p_1^3[M])$$

and

$$\hat{A}(M) = \frac{1}{967680}(-16p_3[M] + 44p_1p_2[M] - 31p_1^3[M])$$

it is simple to check that the linear combination f is given by

$$f([M]) = \lambda_1 \mathrm{sign}(M) + \lambda_2 \hat{A}_3(M) + \lambda_3 p_3[M] \quad \text{for } \lambda_i \in \mathbb{Q}.$$

Moreover, the coefficients of the elliptic genus in dimension 12 are spanned by the signature and the $\hat{A}$-genus, and so we conclude that $\lambda_3 \neq 0$.

We now evaluate the linear combination f on the family X_c^{12}. We recall that the latter is non-negatively curved and spin when c is even. We also note that by multiplicativity, the elliptic genus vanishes on X_{2c}^{12}. By Lemma 2.2, we conclude that

$$f([X_{2c}^{12}]) = \lambda_3 p_3[X_{2c}^{12}] = -8\lambda_3 c^3$$

and f is unbounded. Thus, we have shown that f is unbounded on non-negatively curved spin manifolds, which is exactly the claim. □

Proof of Theorem 1.3 for $k = 4$. The proof goes along the same lines as the previous case. In this dimension, the coefficients of the elliptic genus are spanned by the signature, the $\hat{A}$-genus and the index of the twisted Dirac operator $\hat{A}(M; T_{\mathbb{C}}M)$. Moreover,

$\Omega_{16}^{\mathrm{SO}} \otimes \mathbb{Q}$ is a five-dimensional $\mathbb{Q}$-vector space and f is expressed by a linear combination

$$f([M]) = \lambda_1 \mathrm{sign}(M) + \lambda_2 \hat{A}_4(M) + \lambda_3 \hat{A}_4(M; T_{\mathbb{C}}M) + \lambda_4 p_1^4[M] + \lambda_5 p_4[M]$$

for some $\lambda_i \in \mathbb{Q}$. By the multiplicativity in spin fibre bundles, the elliptic genus vanishes on Y_c^{16} and, in view of Lemma 2.3, the evaluation yields

$$f([Y_c^{16}]) = c^3(768(12 + 56c^2)\lambda_4 + 288\lambda_5).$$

Clearly, this is unbounded for $(\lambda_4, \lambda_5) \neq (0, 0)$, and the claim follows. □

Proof of Theorem 1.3 for $k = 5$. In this dimension, a linear combination of Pontryagin numbers $f : \Omega_{20}^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$ has the form

$$\begin{aligned} f([M]) = \lambda_1 \mathrm{sign}(M) + \lambda_2 \hat{A}_5(M) + \lambda_3 \hat{A}_5(M; T_{\mathbb{C}}M) + \lambda_4 p_5[M] \\ + \lambda_5 p_1 p_4[M] + \lambda_6 p_2 p_3[M] + \lambda_7 p_1^2 p_3[M] \end{aligned}$$

with coefficients $\lambda_i \in \mathbb{Q}$. Evaluating on Z_{2c}^{20}, we obtain by Lemma 2.4

$$\begin{aligned} f([Z_{2c}^{20}]) = 2^7 \cdot c^7(-3\lambda_6 - 9\lambda_7) + 2^5 \cdot c^5(-24\lambda_5 - 96\lambda_6 - 240\lambda_7) \\ + 2^3 \cdot c^3(-28\lambda_4 - 232\lambda_5 - 580\lambda_6 - 1056\lambda_7). \end{aligned}$$

By the product formula for the Pontryagin classes we compute

$$f([X_{2c}^{12} \times \mathbb{H}\mathrm{P}^2]) = 2^3 \cdot c^3(-7\lambda_4 - 46\lambda_5 - 73\lambda_6 - 108\lambda_7).$$

Both families are spin and non-negatively curved. If $f([Z_{2c}^{20}])$ and $f([X_{2c}^{12} \times \mathbb{H}\mathrm{P}^2])$ are bounded, then all coefficients of these polynomials in c must vanish, which implies $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$. So a bounded linear combination f must be contained in the span of the coefficients of the elliptic genus. □

As a consequence, we are able to prove Theorem 1.1. The proof follows from the polynomial structure of the rational oriented cobordism ring.

Proof of Theorem 1.1. For a given dimension, we need to find a family of non-negatively curved spin manifolds with mutually distinct cobordism types. We recall that the rational cobordism type is uniquely determined by the Pontryagin numbers. In dimensions 12 and 16, such families are given by X_{2c}^{12} and Y_c^{16}, and the claim follows from Theorem 1.3.

For the dimensions $4k \geqslant 20$, the family $X_{2c}^{12} \times \mathbb{H}\mathrm{P}^{k-3}$ has the desired properties. In fact, this family is spin and non-negatively curved via the product metric. On the other hand, it is well known that

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} = \mathbb{Q}[[X_2(4)], [\mathbb{H}\mathrm{P}^2], [\mathbb{H}\mathrm{P}^3], \ldots].$$

In other words, the $K3$ surface $X_2(4)$ and the quaternionic projective spaces form a sequence generating the rational cobordism ring as a polynomial ring. Therefore, multiplication with the element $[\mathbb{H}\mathrm{P}^{k-3}]$ is injective for $k \geqslant 5$. Hence, the cobordism types of $X_{2c}^{12} \times \mathbb{H}\mathrm{P}^{k-3}$ are mutually distinct, which completes the proof. □

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